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AN EXTENSION OF MULTI-VALUED QUASI-GENERALIZED SYSTEM

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Abstract. Recently, Kazmi and Khan [7] introduced a kind of equilibrium problem called *generalized system* (GS) with a single-valued bi-operator F. Next, in [10], the first author considered a generalization of (GS) into a multi-valued circumstance called the multi-valued quasi-generalized system (in short, MQGS). In the current work, we provide an extension of (MQGS) into a system of (MQGS) in general settings. This system is called the *generalized multi-valued quasi-generalized system* (in short, GMQGS). Using the existence theorem for abstract economy by Kim [8], we prove the existence of solutions for (GMQGS) in the framework of Hausdorff topological vector spaces. As an application, an existence result of a system of generalized vector quasi-variational inequalities is derived.

1. Introduction

The equilibrium problem (EP) has been intensively studied, beginning with Blume and Oettli [2] where they proposed it as a generalization of optimization and variational inequality problem. It turns out that this problem includes, as special cases, other problems such as the fixed point and coincidence point problem, the complementarity problem, the Nash equilibrium problem, etc. Its numerous extensions and applications can be found in the literature. See, e.g., [1, 3, 4] and the references therein. Recently, Kazmi and Khan [7] introduced a kind of EP called generalized system (GS) with a single-valued bi-operator F. Their result extends the strong vector variational inequality (SVVI)

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studied in Fang and Huang [6] in real Banach spaces. Next, in [10], the first author introduced a generalization of (GS) into a multi-valued circumstance called the multi-valued quasi-generalized system (in short, MQGS), and based on the existence theorem for 1-person game by Ding, Kim and Tan [5], he established an existence theorem of (MQGS) within the framework of Hausdorff topological vector spaces. Also an existence result of the generalized vector quasi-variational inequality problem was derived.

In this note, as a continuation of the previous works [9, 10], we provide an extension of (MQGS) into a system of (MQGS) in general settings. This system is called the *generalized multi-valued quasi-generalized system* (in short, GMQGS). Using the existence theorem for abstract economy by Kim [8], we prove the existence of solutions for (GMQGS) under the circumstances of Hausdorff topological vector spaces. As an application, an existence result of a system of generalized vector quasi-variational inequalities is derived.

2. Preliminaries

We begin with taking a brief look at several standard definitions and terminologies concerned with multi-valued functions (or multifunctions). Let X, Y be nonempty topological spaces and $T : X \rightrightarrows Y$ be a multifunction. Then $T : X \rightrightarrows Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. For each $y \in Y$, the set $T^{-1}(y) = \{x \in X \mid y \in T(x)\}$ is called *the lower section* of T at y. We denote by $\operatorname{cl} T$ the multifunction $\operatorname{cl} T(x) = \overline{T(x)}$ for all $x \in X$. In the case that X is a nonempty set of a Hausdorff topological vector space E, a multifunction $T : X \rightrightarrows X$ is said to be of *class* L [11] if (i) for each $x \in X$, $x \notin \operatorname{co} T(x)$ (ii) for each $y \in X$, $T^{-1}(y)$ is open in X where $\operatorname{co} T(x)$ stands for the convex hull of $T(x)$. A nonempty subset C of E is called a *convex cone* if $\lambda C \subseteq C$, for all $\lambda > 0$ and $C + C = C$.

From now on, unless otherwise specified, we work under the following settings: Let I be a finite set, that is, $\{1, \dots, n\}$. For each $i \in I$, let X_i, Y_i be Hausdorff topological vector spaces and let K_i be a nonempty convex subset of X_i. Let C_i be a pointed closed convex cone in Y_i (not necessarily, $\operatorname{int} C_i \neq \emptyset$). Let $K := \Pi_{i \in I} K_i$, and for each $i \in I$, let $F_i : K \times K_i \rightrightarrows Y_i$ be a nonempty multi-valued bi-operator such that $F_i(x, y) \supseteq \{0\}$ for each $x = (x_i)_{i \in I} \in K,\ y \in K_i$. Let $A_i : K \rightrightarrows K_i$

be a multifunction. Now we define the *generalized multi-valued quasi-generalized system* (GMQGS) to be the problem of finding $x \in K$ such that for each $i \in I$, $x_i \in \operatorname{cl} A_i(x)$ and,

$$F_i(x,y) \nsubseteq -C_i \setminus \{0\}, \quad \text{for all } \ y \in A_i(x). \qquad (GMQGS)$$

When I is singleton, (GMQGS) reduces to (MQGS) [10], and when I is singleton and $A_i(x) \equiv K_i$ for each $x \in K$, (MQGS) becomes the multi-valued generalized system(MGS) [9].

A multifunction $G_i : K_i \rightrightarrows Y_i$ is said to be C_i-*convex* if $\forall x, y \in K_i$, $\forall \lambda \in [0,1]$,

$$G_i(\lambda x + (1-\lambda)y) \subseteq \lambda G_i(x) + (1-\lambda)G_i(y) - C_i.$$

One more terminology is necessary. An abstract economy (or a generalized game) $\Gamma = (K_i, A_i, P_i)_{i\in I}$ is defined as a family of ordered triples (K_i, A_i, P_i) where $P_i : K \rightrightarrows K_i$ is also a multifunction. Moreover, $P_i^{'} : K \rightrightarrows K$ denotes the multifunction defined by $P_i^{'}(x) = \{y \in K \mid y_i \in P_i(x)\}$. As mentioned in the introduction, the following lemma is a basic tool to obtain the main result of current work.

Lemma 2.1. (Kim [8, Corollary 2]) *Let $\Gamma = (K_i, A_i, P_i)_{i\in I}$ be an abstract economy satisfying the following conditions: for each $i \in I$,*
(1) K_i is a nonempty compact convex subset of a Hausdorff topological vector space, and let $K := \Pi_{i\in I} K_i$;
(2) for each $x \in K$, $A_i(x)$ is nonempty convex;
(3) $A_i : K \rightrightarrows K_i$ has open lower sections;
(4) $\operatorname{cl} A_i : K \rightrightarrows K_i$ is upper semicontinuous;
(5) for each $i \in I$, $A_i^{'} \cap P_i^{'} : K \rightrightarrows K$ is of class L.
Then Γ has an equilibrium point $\hat{x} = (\hat{x}_i)_{i\in I} \in K$, i.e., for each $i \in I$, $\hat{x}_i \in \operatorname{cl} A_i(\hat{x})$, and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

3. Main result

We begin with the following simple observation:

Lemma 3.1. *Let A and B be nonempty subsets of two vector spaces E and F, respectively. Then we have*

$$co\,(A \times B) = co\,A \times co\,B.$$

Proof. $\operatorname{co}(A \times B) \subseteq \operatorname{co} A \times \operatorname{co} B$ is easy. For the converse, take any $(x,y) \in \operatorname{co} A \times \operatorname{co} B$. Then $x = \sum_{i=1}^{n} \lambda_i a_i$ and $y = \sum_{j=1}^{m} \mu_j b_j$ for some

$a_i \in A$ and $b_j \in B$, and some positive λ_i summing to 1 and μ_j summing to 1. So we get

$$\begin{aligned}(x, y) &= \left(\sum_{i=1}^{n} \lambda_i a_i, y\right) = \sum_{i=1}^{n} \lambda_i (a_i, y) \\ &= \sum_{i=1}^{n} \lambda_i \left(a_i, \sum_{j=1}^{m} \mu_j b_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j (a_i, b_j) \in \operatorname{co}(A \times B)\end{aligned}$$

because $\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j = \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{m} \mu_j) = \sum_{i=1}^{n} \lambda_i = 1$. Hence $\operatorname{co} A \times \operatorname{co} B \subseteq \operatorname{co}(A \times B)$. This completes the proof. □

Lemma 3.2. *For a multifunction $D_i : K \rightrightarrows K_i$, the followings are equivalent:*
(i) $D_i' : K \rightrightarrows K$ is of class L;
(ii) for each $x \in K$, $x_i \notin \operatorname{co} D_i(x)$ and for each $y \in K_i$, $D_i^{-1}(y)$ is open in K.

Proof. First notice that

$$D_i'(x) = K_1 \times \ldots \times D_i(x) \times \ldots \times K_n.$$

Hence by Lemma 3.1, we have

$$\operatorname{co} D_i'(x) = K_1 \times \ldots \times \operatorname{co} D_i(x) \times \ldots \times K_n. \tag{1}$$

Also observe that

$$\forall y \in K,\ (D_i')^{-1}(y) = \{x \in K \mid D_i'(x) \ni y\} = \{x \in K \mid D_i(x) \ni y_i\}. \tag{2}$$

Then the proof is immediate from (1) and (2). □

Now we are in a position to state our main result.

Theorem 3.3. *For each $i \in I$, let K_i be a nonempty compact convex subset of a Hausdorff topological vector space X_i, and $K = \Pi_{i \in I} K_i$. Let $F_i : K \times K_i \rightrightarrows Y_i$ be C_i-convex in the second variable, and let $A_i : K \rightrightarrows K_i$ be a multifunction such that $\operatorname{cl} A_i$ is upper semicontinuous, $A_i(x)$ is nonempty convex for all $x \in K$. Assume that*
(1) for each $y \in K_i$, $\{x \in K \mid F_i(x, y) \subseteq -C_i \setminus \{0\}\}$ is open in K;
(2) for each $y \in K_i$, $A_i^{-1}(y)$ is open in K.
Then (GMQGS) is solvable.

Proof. For each $i \in I$, we first define a multifunction $P_i : K \rightrightarrows K_i$ by

$$P_i(x) := \{y \in K_i \mid F_i(x, y) \subseteq -C_i \setminus \{0\}\}, \text{ for each } x \in K.$$

Then we have
(i) For each $x = (x_i)_{i \in I} \in K$, $x_i \notin \operatorname{co} P_i(x)$. Indeed, suppose the contrary, i.e., there exists $x \in K$ such that $x_i \in \operatorname{co} P_i(x)$. Then there exist $\{y_1, \cdots, y_n\} \subseteq P_i(x)$ and $0 < \lambda_1, \ldots, \lambda_n < 1$ such that $\sum_{j=1}^n \lambda_j = 1$ and $x_i = \sum_{j=1}^n \lambda_j y_j$. Hence, for each $j = 1, \ldots n$,

$$F_i(x, y_j) \subseteq -C_i \setminus \{0\}. \tag{3}$$

Since $F_i(x, \cdot)$ is C_i-convex, by Kum [10, Lemma 3.1] and (3), we have

$$\begin{aligned}
\{0\} \subseteq F_i(x, x_i) = F_i &\left(x, \sum_{j=1}^n \lambda_j y_j \right) \\
&\subseteq \lambda_1 F_i(x, y_1) + \cdots + \lambda_n F_i(x, y_n) - C_i \\
&\subseteq -\big[\lambda_1 (C_i \setminus \{0\}) + \cdots + \lambda_n (C_i \setminus \{0\}) + C_i\big] \\
&\subseteq (-C_i \setminus \{0\}) - C_i = -C_i \setminus \{0\},
\end{aligned}$$

which is a contradiction. Thus, for all $x \in K$, $x_i \notin \operatorname{co} P_i(x)$ so that $x_i \notin \operatorname{co}(A_i \cap P_i)(x)$, $\forall x \in K$.

(ii) For each $y \in K_i$,

$$\begin{aligned}
(A_i \cap P_i)^{-1}(y) &= \{x \in K \mid y \in A_i(x) \text{ and } F_i(x, y) \subseteq -C_i \setminus \{0\}\} \\
&= \{x \in K \mid F_i(x, y) \subseteq -C_i \setminus \{0\}\} \cap A_i^{-1}(y)
\end{aligned}$$

By the assumptions (1) and (2), $(A_i \cap P_i)^{-1}(y)$ is open in K for each $y \in K_i$.
(iii) Conclusions of (i) and (ii) is nothing but the fact that $(A_i \cap P_i)' = A_i' \cap P_i'$ is of class L by means of Lemma 3.2. Therefore, the abstract economy $\Gamma = (K_i, A_i, P_i)_{i \in I}$ satisfies all the conditions of Lemma 2.1 so that there is an equilibrium $\hat{x} \in K$ such that for each $i \in I$, $\hat{x}_i \in \operatorname{cl} A_i(\hat{x})$, and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$, that is,

$$F_i(\hat{x}, y) \nsubseteq -C_i \setminus \{0\} \quad \text{for all} \quad y \in A_i(\hat{x}).$$

This completes the proof. □

Remark 3.4. Theorem 3.3 is an extension of Kum [10, Theorem 3.2] for n-variables. It reduces to Kum [10, Theorem 3.2] when I is a singleton.

As an application of Theorem 3.3, we derive an existence result of a system of generalized vector quasi-variational inequalities as follows.

COROLLARY 3.5. *Let A_i be the same as in Theorem 1. For each $i \in I$, let $T_i : K \rightrightarrows L(X_i, Y_i)$ be a multifunction such that for each $y \in K_i$, the set $\{x \in K \mid \langle T_i(x), y - x_i\rangle \subseteq -C_i \setminus \{0\}\}$ is open in K where $L(X_i, Y_i)$ denotes the space of all continuous linear operators from X_i to Y_i. Then there exists $\hat{x} \in K$ such that $\forall i \in I,\ \hat{x} \in \mathrm{cl}\, A_i(\hat{x})$ and*

$$\langle T_i(\hat{x}), y - \hat{x}_i\rangle \nsubseteq -C_i \setminus \{0\} \quad \text{for all} \quad y \in A_i(\hat{x}).$$

Proof. For $x \in K$ and $y \in K_i$, define the bi-operator $F_i : K \times K_i \rightrightarrows Y_i$ by $F_i(x, y) = \langle T_i(x), y - x_i\rangle$. To apply Theorem 1, it suffices to check that F_i is C_i-convex in the second variable. In fact, fix $x \in K$. For $\forall y_1, y_2 \in K_i,\ \forall \lambda \in [0, 1]$, we have

$$\begin{aligned} F_i(x, \lambda y_1 + (1-\lambda)y_2) &= \langle T_i(x),\, \lambda(y_1 - x_i) + (1-\lambda)(y_2 - x_i)\rangle \\ &\subseteq \lambda\langle T_i(x),\, y_1 - x_i\rangle + (1-\lambda)\langle T_i(x),\, y_2 - x_i\rangle - \{0\} \\ &\subseteq \lambda\langle T_i(x),\, y_1 - x_i\rangle + (1-\lambda)\langle T_i(x),\, y_2 - x_i\rangle - C_i \\ &= \lambda F_i(x, y_1) + (1-\lambda)F_i(x, y_2) - C_i. \end{aligned}$$

By Theorem 1, there exists $\hat{x} \in K$ such that $\forall i \in I,\ \hat{x}_i \in \mathrm{cl}\, A_i(\hat{x})$ and

$$F_i(\hat{x}, y) \nsubseteq -C_i \setminus \{0\} \quad \text{for all} \quad y \in A_i(\hat{x}),$$

which implies that

$$\langle T_i(\hat{x}), y - \hat{x}_i\rangle \nsubseteq -C_i \setminus \{0\} \quad \text{for all} \quad y \in A_i(\hat{x}).$$

□

REMARK 3.6. Corollary 3.5 is a general version of Kum [10, Corollary 3.4] for n-variables. It reduces to Kum [10, Corollary 3.4] when I is a singleton.

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